Rational Approximations and Analytic Semigroups

András Sike

Operator Semigroups in Numerical Analysis seminar talk

Eötvös Loránd Tudományegyetem

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EÖTVÖS LORÁND
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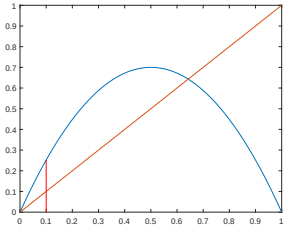
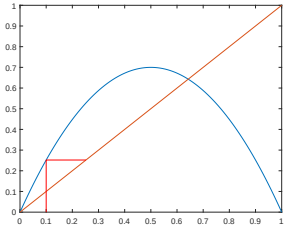


Motivation

We have seen that rational approximations behave nicely if the generator is selfadjoint:

- We have already established stability and convergence results in the scalar case;
- Selfadjoint operators can be considered as multiplication operators, therefore by imposing some geometric conditions on the spectrum, the scalar case can be extended.



Sectorial Operators

Definition. Let $(A, \mathcal{D}(A))$ be a linear operator on some Banach space X, with angle $\delta \in (0, \frac{\pi}{2})$ given. The operator is called *sectorial of angle* α, if the sector

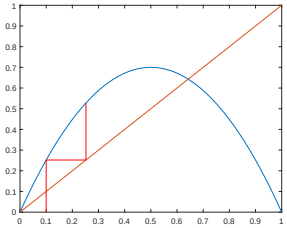
$$\Sigma_{\frac{\pi}{2}+\delta} := \left\{ \lambda \in \mathbb{C} \setminus 0 : |\arg(\lambda)| < \frac{\pi}{2} + \delta \right\}$$

is contained in $\varrho(A)$ resolvent set, and the following estimate holds true:

$$\sup_{\lambda \in \Sigma_{\frac{\pi}{2}+\delta'}} \|\lambda \mathcal{R}(\lambda, A)\| < \infty \qquad \forall\, \delta' \in (0, \delta).$$

If A is densely defined, and γ is a suitable enough curve, we can define the (strongly continuous, analytic) semigroup

$$T(z) := e^{zA} := \frac{1}{2\pi i} \int_\gamma e^{\lambda z} \mathcal{R}(\lambda, A)\, \mathrm{d}\lambda, \qquad \forall\, z \in \Sigma_\delta.$$



Admissible curves

Suppose we are given an operator A, that is sectorial of angle $\delta \in (0, \frac{\pi}{2})$ and an angle $\theta \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$. Let us denote the sector

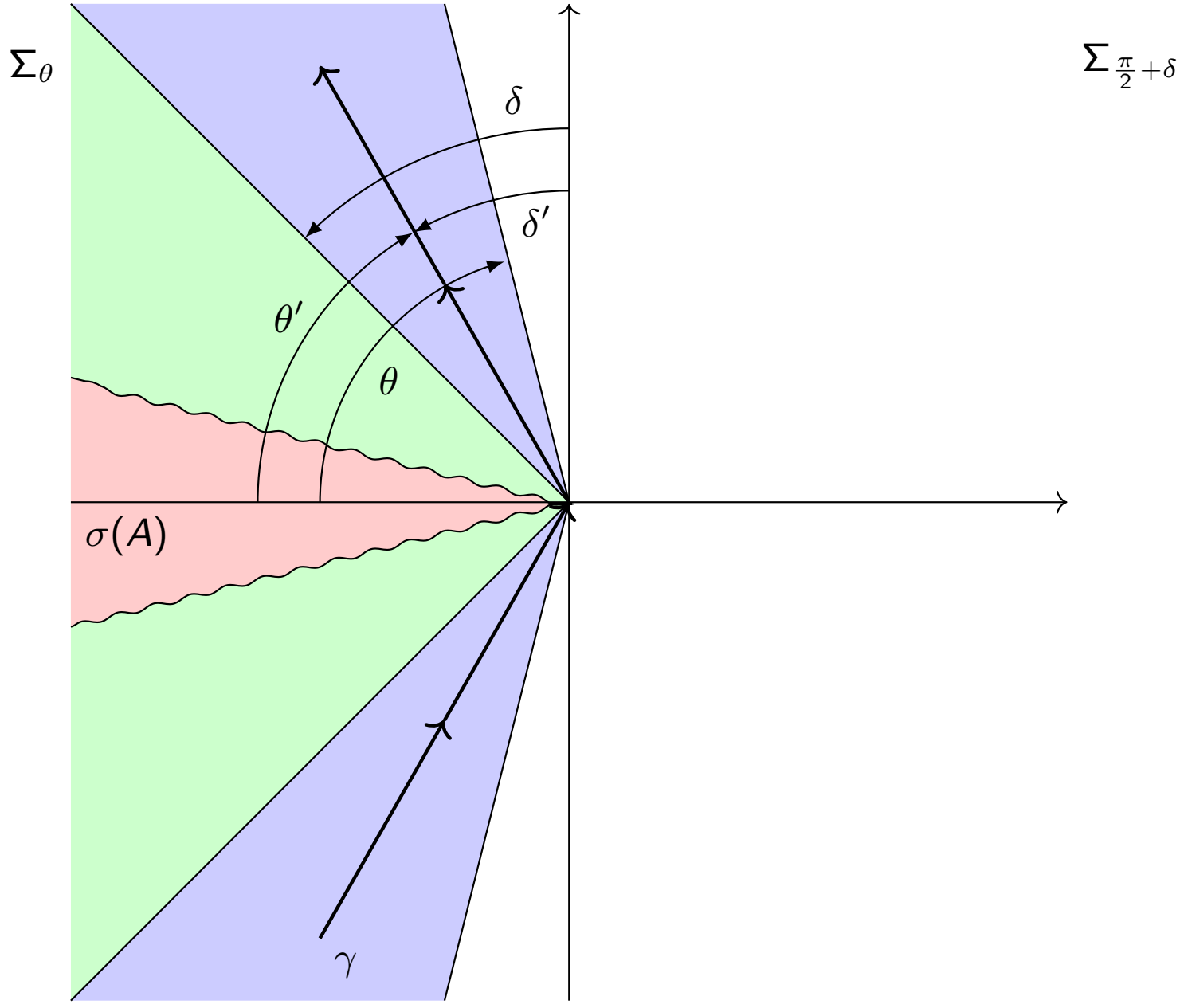
$$\Sigma_\theta := \{z \in \mathbb{C} \setminus \{0\} : |\arg(-z)| < \theta\} = -\Sigma_\theta = \mathbb{C} \setminus \overline{\Sigma_{\pi-\theta}}$$

and take the set of functions

$$\mathcal{H}_0^\infty(\Sigma_\theta) := \left\{ F : \Sigma_\theta \to \mathbb{C} : F \text{ holomorphic}, \exists\, \varepsilon > 0, C \geq 0 : |F(z)| \leq \frac{C|z|^\varepsilon}{(1+|z|)^{2\varepsilon}} \forall\, z \in \Sigma_\theta \right\}$$

Similarly to the semigroup, the effect of F on A will be defined via line integrals. We need curves to integrate along, so we will consider a curve admissible if it takes the form $\gamma_{\delta'} = -\gamma^{(2)}_{\delta'} + \gamma^{(1)}_{\delta'}$, where

$$\gamma^{(1)}_{\delta'} := se^{i(\frac{pi}{2}+\delta')}, \quad \gamma^{(2)}_{\delta'} := se^{-i(\frac{pi}{2}+\delta')} \qquad \forall\, \delta' \in (\frac{pi}{2} - \theta, \delta),\ s \in (0, \infty).$$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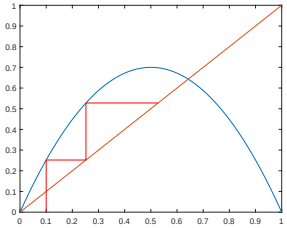
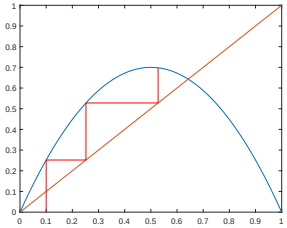


Effect of F on the Generator A

Definition. If A is a sectorial operator of angle $\delta > 0$, and $\theta \in (\frac{\pi}{2} - \delta, \frac{\pi}{2}]$, with $\gamma = \gamma_{\delta'}$ admissible curve given, then for any $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$ we define

$$F(A) := \Phi_A(F) := \frac{1}{2\pi i} \int_\gamma F(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda.$$

- $F(A) \in \mathcal{L}(X)$, since the integral is absolutely convergent (A is sectorial, F decays fast enough);
- The integral is well defined and yields the same value for any admissible curve $\gamma_{\delta'}$;
- $\mathcal{H}_0^\infty(\Sigma_\theta)$ is and algebra with respect to the pointwise operators.



A Basic Proposition

Proposition. The following three statements are true:

a) The mapping

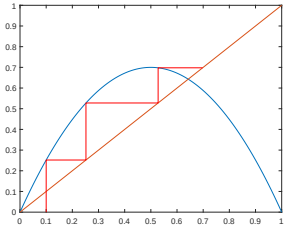
$$\Phi_A : \mathcal{H}_0^\infty(\Sigma_\theta) \to \mathcal{L}(X)$$

is an algebra homomorphism;

b) For any given $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$, and a closed operator B that commutes with the resolvent of A, B also commutes with $F(A)$;

c) For all $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$, $\mu \in \mathbb{C} \setminus -\overline{\Sigma_\theta} = \Sigma_{\pi-\delta}$ we have the following:

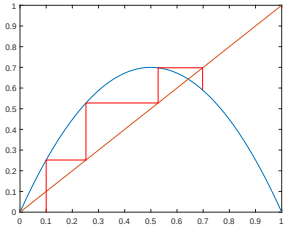
$$G(z) := (\mu - z)^{-1} F(z) \Rightarrow G \in \mathcal{H}_0^\infty(\Sigma_\theta),\ G(A) = \mathcal{R}(\mu, A)F(A).$$



Proof of **a)** and **b)**

Proof.

- For **a)**, we need to check linearity and multiplicativity;
- Linearity follows from the fact that we defined it as composition of linear operators;
- It is technical, but not difficult to show multiplicativity via using the resolvent identity;
- The proof of **b)** is omitted.

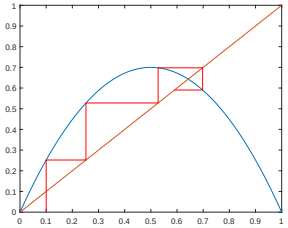


Proof of **c)**

Fix $\mu \in \mathbb{C} \setminus -\overline{\Sigma_\theta}$ and γ admissible curve. Then we can write

$$\begin{aligned}
\mathcal{R}(\mu, A)F(A) &= \frac{1}{2\pi i}\int_\gamma F(\lambda)\mathcal{R}(\mu, A)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda = \\
&= \frac{1}{2\pi i}\int_\gamma F(\lambda)\,(\mu-\lambda)^{-1}\,(\mathcal{R}(\lambda, A) - \mathcal{R}(\mu, A))\,\mathrm{d}\lambda = \\
&= \frac{1}{2\pi i}\int_\gamma F(\lambda)\,(\mu-\lambda)^{-1}\,\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda - \frac{1}{2\pi i}\int_\gamma F(\lambda)\,(\mu-\lambda)^{-1}\,\mathcal{R}(\mu, A)\,\mathrm{d}\lambda \\
&= G(A) + 0,
\end{aligned}$$

by the Cauchy theorem. ■



Extension

We can notice that the function corresponding to the implicit Euler method, $F(z) = \frac{1}{1-z}$ is not a member of the set $\mathcal{H}_0^\infty(\Sigma_\theta)$. To make our theorems valid for actual practical methods, we define the set

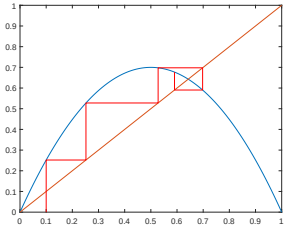
$$\mathcal{E}(\Sigma_\theta) := \mathcal{H}_0^\infty(\Sigma_\theta) + \operatorname{lin}\{1\} + \operatorname{lin}\left\{(1-z)^{-1}\right\}.$$

Lemma. The following three statements are true:

a) The sum defining $\mathcal{E}(\Sigma_\theta)$ is a direct sum;

b) The linear space $\mathcal{E}(\Sigma_\theta)$ is an algebra;

c) If $F \in \mathcal{E}(\Sigma_\theta)$, then we also have $G(z) := F\left(\frac{1}{z}\right) \in \mathcal{E}(\Sigma_\theta)$.



Proof of **a)**

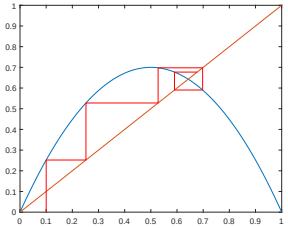
Proof. Let $F \in \mathcal{E}(\Sigma_\theta)$ and define the (existing) limits

$$c := \lim_{z \to 0, z \in \Sigma_\theta} F(z), \qquad d := \lim_{z \to \infty, z \in \Sigma_\theta} F(z),$$

and we can then write

$$\begin{aligned} F(z) &= \left(F(z) - d\mathbf{1} + (d-c)\frac{1}{1-z} \right) + d\mathbf{1} - (d-c)\frac{1}{1-z} = \\ &= G(z) + d1 = \mathbf{1} - (d-c)\frac{1}{1-z}, \end{aligned}$$

with $G \in \mathcal{H}_0^\infty(\Sigma_0^\infty)$ holding true.

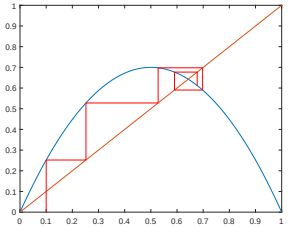


Proof of **b)** and **c)**

It is sufficient to show that for arbitrary $F \in \mathcal{H}_0^\infty(\Sigma_\theta)$ and $G(z) = \dfrac{1}{1-z}$ the functions FG and G^2 are contained in $\mathcal{E}(\Sigma_\theta)$. By definition, $FG \in \mathcal{H}_0^\infty(\Sigma_\theta) \subset \mathcal{E}(\Sigma_\theta)$, and for G^2 we have

$$G^2(z) = \frac{1}{(1-z)^2} = \frac{1}{1-z} + \frac{z}{(1-z)^2},$$

where the second function is a member of the set $\mathcal{H}_0^\infty(\Sigma_\theta)$.
Lastly, the proof of **c)** is omitted. ■



Meaning of the Lemma

The first statement of the lemma allows us to extend the functional calculus to the broader set: For any $F \in \mathcal{E}(\Sigma_\theta)$ define

$$F(0) := \lim_{z \to 0, z \in \Sigma_\theta} F(z), \qquad F(\infty) := \lim_{z \to \infty, z \in \Sigma_\theta} F(z).$$

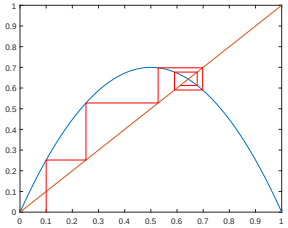
Then, by using $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$, we can write $F(z)$ in the following form:

$$F(z) = G(z) + \frac{F(0) - F(\infty)}{1 - z} + F(\infty)\mathbf{1}.$$

We can also extend the mapping Φ_A in the following manner:

$$F(A) := \Phi_A(F) := \Phi_A(G) + (F(0) - F(\infty))\,\mathcal{R}(1, A) + F(\infty)I.$$

We first establish that this definition is consistent with the one developed for the exponential function.



Consistency of the Extension

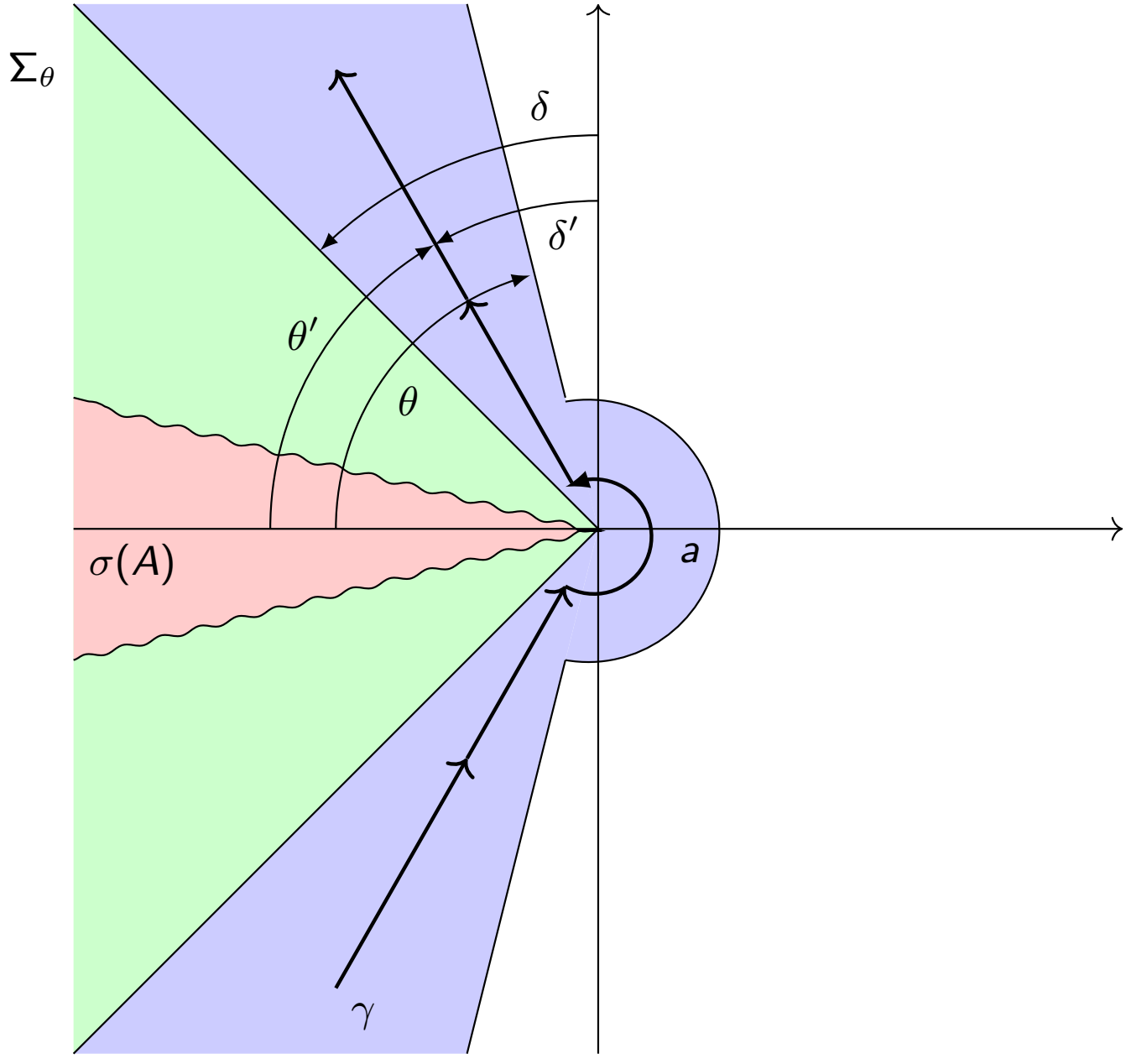
Theorem. Let $F : \Sigma_\theta \to \mathbb{C}$ be a holomorphic function that has a holomorphic extension at point 0, and further, we can choose $C \geq 0, \varepsilon > 0$ such that

$$|F(z)| \leq \frac{C}{1+|z|^\varepsilon} \quad \forall\, z \in \Sigma_\theta.$$

Then $F \in \mathcal{E}(\Sigma_\theta)$, and further, if we have a suitable curve $\gamma = \gamma_{\delta',a}$ (cf. next slide), then

$$\Phi_A(F) = \frac{1}{2\pi i}\int_\gamma F(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda.$$

Even further, we also have that for any $\mu \in \mathbb{C} \setminus -\overline{\Sigma_\theta}$ and $F(z) := (\mu - z)^{-k}$ with $k \in \mathbb{N}$, we have $\Phi_A(F) = \mathcal{R}(\mu, A)^k$.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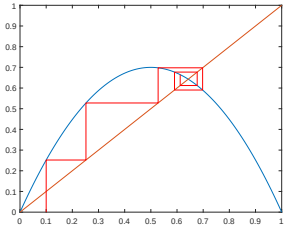


Proof I

Proof. Note that with appropiate $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$ we have

$$F(z) = F(z) - \frac{F(0)}{1-z} + \frac{F(0)}{1-z} = G(z) + \frac{F(0)}{1-z},$$

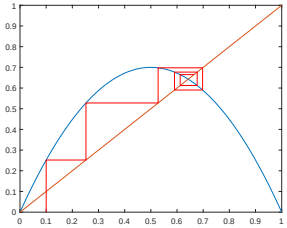
since G holomorphic and takes value of 0 at 0, and so $|G(z)| < C|z|$ in a suitable neighborhood of 0. As we approach ∞, the estimate remains valid, meaning that $F \in \mathcal{E}(\Sigma_\theta)$.

We need to show the convergence of the integral. In finite points it is trivial, at infinity, take

$$\frac{1}{2\pi i}\int_{\gamma_{\delta',a}} G(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda.$$

Using that $G \in \mathcal{H}_0^\infty(\Sigma_\theta)$, we can freely let $a \to 0$ without changing the value of the integral:

$$\frac{1}{2\pi i}\int_{\gamma} G(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda = \lim_{a\to\infty}\frac{1}{2\pi i}\int_{\gamma_{\delta',a}} G(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda = \Phi_A(G).$$



Proof II

If we can show

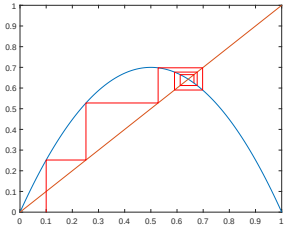
$$\mathcal{R}(\mu, A)^k = \frac{1}{2\pi i} \int_\gamma \frac{1}{(\mu - \lambda)^k} \mathcal{R}(\lambda, A)\, \mathrm{d}\lambda,$$

then the proof is finished. To this end, take the curve $\widehat{\gamma} := -\gamma_{\mu, a+|\mu|}$ and use the Cauchy theorem:

$$\int_{\widehat{\gamma}} \frac{\mathcal{R}(\lambda, A)^k}{\mu - \lambda}\, \mathrm{d}\lambda = 0.$$

Henceforth, we can use Cauchy's formula for the derivative and obtain

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{\mathcal{R}(\lambda, A)}{(\mu - \lambda)^k}\, \mathrm{d}\lambda &= \frac{1}{2\pi i} \int_\gamma \frac{\mathcal{R}(\lambda, A)^k}{\mu - \lambda}\, \mathrm{d}\lambda + \frac{1}{2\pi i} \int_{\widehat{\gamma}} \frac{\mathcal{R}(\lambda, A)}{(1 - \lambda)^k}\, \mathrm{d}\lambda = \\ &= \frac{1}{2\pi i} \int_{\gamma + \widehat{\gamma}} \frac{\mathcal{R}(\lambda, A)}{(\mu - \lambda)^k}\, \mathrm{d}\lambda = -(-1)^k \frac{\partial^{k-1}}{\partial^{k-1} z} \mathcal{R}(\mu, A) = \\ &= \mathcal{R}(\mu, A)^k. \quad \blacksquare \end{aligned}$$



Two Further Properties

Proposition. The following all hold true:

a) The mapping $\Phi_A : \mathcal{E}(\Sigma_\theta) \to \mathcal{L}(X)$ is a unital algebra homomorphism;

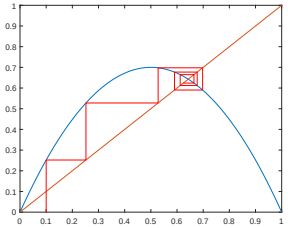
b) If a closed operator B commutes with the resolvent of A, then it also commutes with $F(A)$;

c) For $F(z) = (1-z)^{-1}$ we have

$$F(A) = A\mathcal{R}(1,A) = \mathcal{R}(1,A) - I.$$

Proposition. Let $F : \Sigma_\theta \to \mathbb{C}$ be a holomorphic function on $\overline{\mathbb{C}}$. Then $F \in \mathcal{E}(\Sigma_\theta)$ and for sufficient numbers a and b, and curve γ we have

$$\Phi_A(F) = F(\infty) + \frac{1}{2\pi i}\int_F (\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda.$$



Exponential Functions

If A is a sectorial operator of angle δ, then so is hA for any $h \geq 0$:

$$\|\mathcal{R}(\lambda, hA)\| = \frac{1}{h}\left\|\mathcal{R}\left(\frac{\lambda}{h}, A\right)\right\| \leq \frac{M}{|\lambda|}.$$

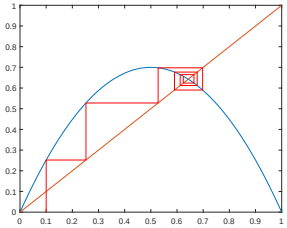
If we take any $\theta \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$, the exponential is in $\mathcal{E}(\Sigma_\theta)$, and so

$$\Phi_{hA}(\exp) = e^{hA}.$$

By an earlier proposition, this is just

$$e^{hA} = \frac{1}{2\pi i}\int_\gamma e^{h\lambda}\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda,$$

with a suitable γ curve.



Rational Functions

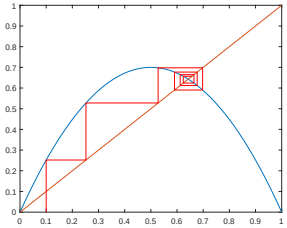
Take r, $A(\alpha)$-stable rational function. Then for any $\theta \in (0, \alpha]$ $r \in \mathcal{E}(\Sigma_\theta)$.

If A is a sectorial operator of angle $\delta \in (\frac{\pi}{2} - \alpha, \frac{\pi}{2})$, then so is hA.

Now we can evaluate $\Phi_A(r)$ and compare it to $r(hA)$. If we use the same proposition, we get

$$r(A) = r(\infty) + \frac{1}{2\pi i} \int_\gamma (r(\lambda) - r(\infty)) \mathcal{R}(\lambda, A) \, \mathrm{d}\lambda,$$

if γ is the same admissible curve.



Fractional Powers

Fix $\beta > 0$ and $k \in \mathbb{N}$ such that $k \geq \beta$. Consider the function

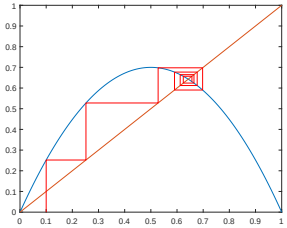
$$F_{\beta,k}(z) := \frac{(-z)^\beta}{(1-z)^k}.$$

With this setting, $F_{\beta,k} \in \mathcal{H}_0^\infty(\Sigma_\theta)$, and so the following definition makes sense:

$$(-A)^\beta := (I-A)^k F_{\beta,k}(A),$$

and the domain of this operator is

$$\mathcal{D}\left((-A)^\beta\right) = \left\{f \in X : F_{\beta,k}(A)f \in \mathcal{D}\left(A^k\right)\right\}.$$



An Important Result

Proposition.

a) The definition of $(-A)^\beta$ is independent of the choice of $k \in \mathbb{N}$;

b) For $h \geq 0$ we have $(-hA)^\beta = h^\beta (-A)^\beta$;

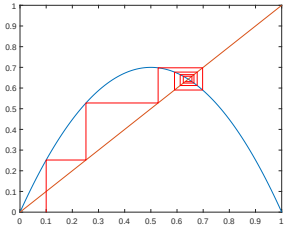
c) For $\eta \in \mathbb{N}$ we have that $(-A)^\beta$ is the β-th power of $-A$;

d) If $0 \in \varrho(A)$, then we have

$$(I - A)^k F_{\beta,k}(A) = \left(\frac{1}{2\pi i} \int_\gamma (-\lambda)^{-\beta} \mathcal{R}(\lambda, A)\, \mathrm{d}\lambda \right)^{-1},$$

where γ is an admissible curve.

Proof. Omitted. ■



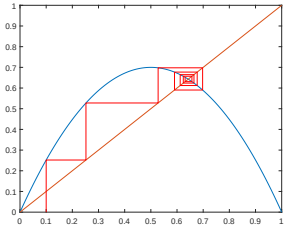
The First Stability Theorem

Theorem. Suppose A is a generator of an analytic semigroup, that is we have

$$\|\mathcal{R}(\lambda, A)\| \leq \frac{M}{|\lambda|} \qquad \text{on a sector } \Sigma_{\frac{\pi}{2}-\delta}.$$

Let r be an $A(\alpha)$-stable rational approximation with some angle $\pi - \delta < \alpha \leq \frac{\pi}{2}$ and $|r(z)| \leq 1$ for any z with $|\arg(z) - \pi| \leq \alpha$, further, suppose $|r(\infty)| < 1$. Then there exists $K \geq 1$ such that

$$\|r(hA)^n\| \leq K \qquad \forall\ h > 0, n \geq 1.$$



Proof I

Proof. We integrate on the curve

$$\left\{|\arg(z)| = \frac{\pi}{2} + \delta, |z| \geq \frac{1}{n}\right\} \cup \left\{|\arg(z)| \leq \frac{\pi}{2} + \delta, |z| = \frac{1}{n}\right\}.$$

We have seen that

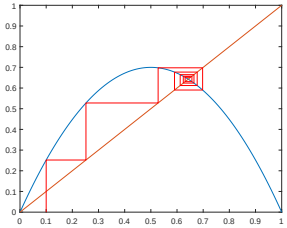
$$r(hA)^n - r(\infty)^n = \frac{1}{2\pi i} \int_\gamma (r(\lambda)^n - r(\infty)^n)\, \mathcal{R}(\lambda, hA)\, \mathrm{d}\lambda.$$

We decompose the integral into four parts. Since r approximates the exponential function, we know that

$$\exists\ \varrho, C, c > 0 : \ |r(z)| \leq |1 + Cz| \leq e^{-c|z|}$$

if $|z| < \varrho$ and $\Re(z) < 0$.
Choose n such that $n \geq \frac{1}{\varrho}$; since $|r(\infty)|$, we also have

$$\exists\ c_2 > 0, 0 < r_0 < 1 : \quad \forall\ \Re(z) < -c_2, |\arg(z) - \pi| \leq \alpha : \ |r(z)| \leq r_0.$$



Proof II

a) $|r(z)| \leq (1 + C|z|)^n \leq e^C$ if $|z| = \frac{1}{n}$;

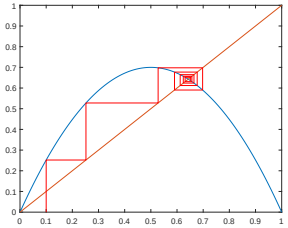
b) $|r(z)| \leq e^{-c|z|}$ if $-\varrho \sin(\delta) \leq \Re(z) \leq -\frac{1}{n}\sin(\delta)$;

c) $|r(z)| \leq 1$ if $-c_2 \leq \Re(z) \leq 0\varrho \sin(\delta)$;

d) $|r(z)| \leq r_0$ if $\Re(z) < -c_2$.

a) and **c)** are bounded curves, and we integrate a bounded function on them. Only the integrals along **b)** and **d)** are interesting.
We can also observe, that on **d)**, we can apply a telescopic identity to obtain

$$|r(z)^n - r(\infty)^n| = |r(z) - r(\infty)| \cdot \left| r(z)^{n-1} + \ldots + r(\infty)^{n-1} \right| \leq \frac{Cnr_0^{n-1}}{|z|}.$$



Proof III

Then, the contribution to the integral along path **d)** is

$$\int_{\gamma\big|_{\Re(\lambda)\le c_2}} |r(\lambda)^n - r(\infty)^n| \cdot \|\mathcal{R}(\lambda, A)\| \, |\mathrm{d}\lambda| \le 2 \int_{\frac{c_2}{\sin(\delta)}}^{\infty} \frac{C n r_0^{n-1}}{s} \frac{M}{s} \, \mathrm{d}s \le \frac{2CMn r_0^{n-1}}{c_2} \sin(\delta).$$

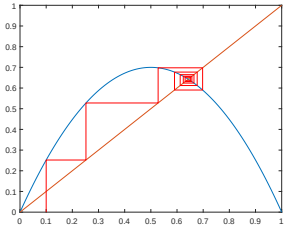
Lastly, we have to check the integrand along **b)**:

$$\int_{-\varrho \sin(\delta) \le \Re(z) \le -\frac{1}{n}\sin(\delta)} M \frac{e^{-Cn|z|}}{|z|} \, |\mathrm{d}z| \le 2M \int_1^{\infty} \frac{e^{-Cs}}{s} \, \mathrm{d}s \le K',$$

and secondly

$$\int_{-\varrho \sin(\delta) \le \Re(z) \le -\frac{1}{n}\sin(\delta)} |r(\infty)|^n \, \frac{M}{|z|} \, |\mathrm{d}z| \le 2M \, |r(\infty)|^n \int_1^{n\varrho} \frac{1}{s} \, \mathrm{d}s = |r(\infty)|^n \, 2M \log(n\varrho) \le K'',$$

so the integral is bounded along all paths. ■



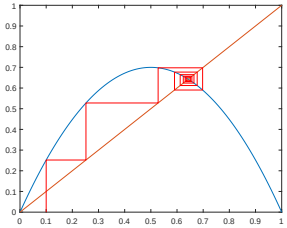
The Second Stability Theorem

Theorem. Suppose A generates an analytic semigroup, $\alpha \in (\pi - \delta, \frac{\pi}{2}]$, r is an $A(\alpha)$-stable rational approximation. Then there exists $K \geq 1$ such that

$$\|r(hA)^n\| \leq K \qquad \forall\ h > 0, n \geq 1.$$

Proof. Omitted.

Remark. This means that the condition $|r(\infty)| < 1$ can be omitted.

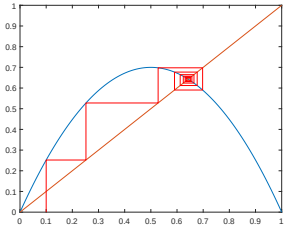


The First Convergence Theorem

Theorem. Let A be a sectorial operator of angle δ and r be an $A(\alpha)$-stable rational approximation of order p, with angle $\alpha \in (\frac{\pi}{2} - \delta, \frac{\pi}{2}]$, and further, suppose $|r(\infty)| < 1$. Then there exists $K > 0$, such that for $t = nh$ we have

$$\left\| r(hA)^n - e^{tA} \right\| \leq K \frac{h^p}{t^p} = \frac{K}{n^p} \qquad \forall\ n \in \mathbb{N}, t \geq 0.$$

Remark. This means that the rational approximation is convergent in the operator norm.



Proof I

Proof. Fix angles $\delta' \in (\frac{\pi}{2} - \delta, \frac{\pi}{2})$, $\theta' := \frac{\pi}{2} - \delta'$ and $\gamma = \gamma_{\delta'}$ admissible curve. We define

$$F_n(z) := r(z)^n - e^{nz},$$

and estimate $\|F_n(hA)\|$. We have $F_n(0) = 0$, and thus, we can rewite as

$$F_n(z) = F_n(z) + F_n(\infty)\frac{z}{1-z} - F_n(z)\frac{z}{1-z} =: G_n(z) - F_n(\infty)\frac{z}{1-z},$$

which means that

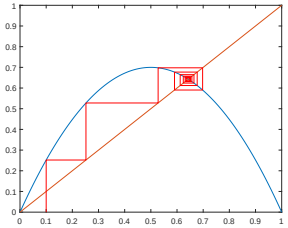
$$F_n(hA) = G_n(hA) - F_n(\infty)A\mathcal{R}(1, A),$$

since

$$\frac{z}{1-z} = A\mathcal{R}(1, A) = A(I - A)^{-1}.$$

We can estimate the norm of the second term, since $|r(\infty)| < 1$ and then by $F_n(\infty) = r(\infty)^n \to 0$ we obtain

$$\|F_n(\infty)A\mathcal{R}(1, A)\| \leq \frac{K'}{n^p} \qquad \forall\ n \in \mathbb{N}.$$



Proof II

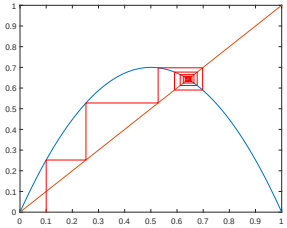
Since $G_n \in \mathcal{H}_0^\infty(\Sigma_\alpha)$ for any $n \in \mathbb{N}$, we can write

$$G_n(hA) = \frac{1}{2\pi i}\int_\gamma G_n(\lambda)\mathcal{R}(\lambda, hA)\,\mathrm{d}\lambda.$$

There is some ball $B_0(\varepsilon_0)$, outside of which

$$\sup\left\{|r(z)| : z \in -\overline{\Sigma_{\theta'}}, |z| \geq \varepsilon_0\right\} =: r_0 < 1.$$

We split γ to its' part lying outside of the ball (γ_-) and inside (γ_+).



Proof III

We first estimate the integral along γ_-. On this domain, we can observe that there is a constant $C > 0$ where

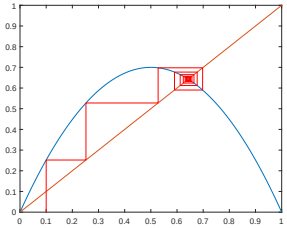
$$|r(z) - r(\infty)| \leq \frac{d}{|z|},$$

and by using a telescopic identity, we can get

$$|r(z)^n - r(\infty)^n| \leq |r(z) - r(\infty)| \sum_{j=0}^{n-1} |r(z)|^{n-1-j} |r(\infty)|^j \leq \frac{1}{|z|} C n r_0^{n-1}.$$

Hence

$$|F_n(z) - F_n(\infty)| \leq |e^{nz}| + |r(z)^n - r(\infty)^n| \leq e^{-n\cos(\alpha)|z|} + \frac{C n r_0^{n-1}}{|z|}.$$



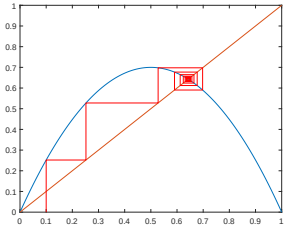
Proof IV

We can then rewrite the integral on γ_- as

$$\left\| \frac{1}{2\pi i} \int_{\gamma_-} G_n(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda \right\| = \left\| \frac{1}{2\pi i} \int_{\gamma_-} \left(F_n(\lambda) - F_n(\infty)\right)\mathcal{R}(\lambda, A) + \frac{F_n(\infty)}{1-\lambda}\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda \right\|$$

$$\leq \frac{M}{\pi} \int_{h_0}^{\infty} \frac{1}{s} \left(e^{-n\cos(\alpha)s} + \frac{1}{s} C n r_0^{n-1} + \frac{1}{s} \left|F_n(\infty)\right| \right) \mathrm{d}s \leq$$

$$\leq \frac{M}{\pi} c' \frac{1}{n^p} + \frac{M}{\pi} c' r_0^{n-1} + \frac{M}{\pi} c' r_0^n \leq \frac{K''}{n^p}.$$



Proof V

Lastly, we need an estimation on γ_+. There are constants $C, c > 0$ where

$$|r(z)^n - e^{nz}| \leq Cn\,|z|^{p+1}\,e^{-nc|z|} \qquad \forall\ n \in \mathbb{N}, z \in -\overline{\Sigma_{\theta'}}, |z| \leq \varepsilon_0.$$

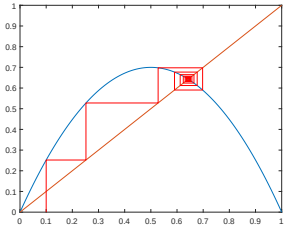
This implies the estimation

$$|G_n(z)| \leq Cn\,|z|^{p+1}\,e^{-nc|z|} + c'\,|z| \cdot |F_n(\infty)|\,,$$

and this can be used to estimate the integral:

$$\begin{aligned}
\left\| \frac{1}{2\pi i} \int_{\gamma_+} G_n(\lambda)\mathcal{R}(\lambda, A)\,\mathrm{d}\lambda \right\| &\leq 2Cmn \int_0^{h_0} s^p e^{-nsc}\,\mathrm{d}s + 2c'h_0\,|F_n(\infty)| \leq \\
&\leq \frac{2CMn}{c^{p+1}n^{p+1}} \int_0^{\infty} t^p e^{-t}\,\mathrm{d}s + 2c'h_0\,|F_n(\infty)| = \\
&= \frac{p!2CM}{c^{p+1}n^p} + 2C'h_0\,|r(\infty)|^n \leq \frac{K'''}{n^p}.
\end{aligned}$$

The proof is finished by unifying the three estimates. ■

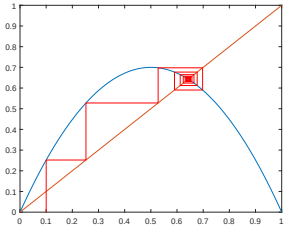


The Second Convergence Theorem

Theorem. Let A be a sectorial operator of angle δ and r be an $A(\alpha)$-stable rational approximation of order p, with angle $\alpha \in (\frac{\pi}{2} - \delta, \frac{\pi}{2}]$, and further, suppose $|r(\infty)| < 1$. Then for any $\beta \in (0, p]$ there is some $K \geq 0$ such that

$$\left\| r(hA)^n f - e^{nhA} f \right\| \leq K h^\beta \left\| (-A)^\beta f \right\| \quad \forall\, f \in \mathcal{D}\left((-A)^\beta \right), h > 0, n \in \mathbb{N}.$$

Remark. This means that the rational approximation is pointwise convergent.



Proof I

Proof. We define

$$F_n(z) := (-z)^{-\beta}\left(r(z)^n - e^{nz}\right), \quad F_n \in \mathcal{H}_0^\infty\left(\Sigma_\alpha\right),$$

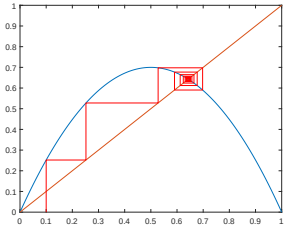
hence we can write

$$F_n(hA) = \frac{1}{2\pi i}\int_\gamma F_n(\lambda)\mathcal{R}(\lambda, hA)\,\mathrm{d}\lambda,$$

and we will again use

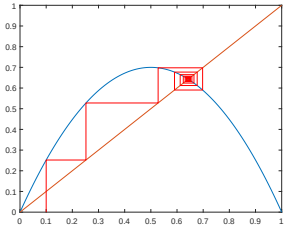
$$|r(z)^n - e^{nz}| \leq Cn\,|z|^{p+1}\,e^{-nc|z|}.$$

We decompose the curve γ into path inside $B_0(1)$ (γ_+) and outside of the ball (γ_-).



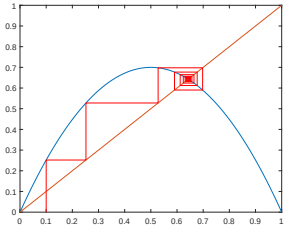
Proof II

$$
\begin{aligned}
\|F_n(hA)\| &\leq \frac{1}{2\pi}\int_\gamma |F_n(\lambda)|\cdot\|\mathcal{R}(\lambda,hA)\|\,|\mathrm{d}\lambda| = \\
&= \frac{1}{2\pi}\int_{\gamma_-} |F_n(\lambda)|\cdot\frac{1}{h}\left\|\mathcal{R}\left(\frac{\lambda}{h},A\right)\right\|\,|\mathrm{d}\lambda| + \\
&\quad + \frac{1}{2\pi}\int_{\gamma_+} |F_n(\lambda)|\cdot\frac{1}{h}\left\|\mathcal{R}\left(\frac{\lambda}{h},A\right)\right\|\,|\mathrm{d}\lambda| \leq \\
&\leq \frac{CM}{2\pi}\int_{\gamma_-} |\lambda|^{p+1-\beta}\, ne^{-cn|\lambda|}\,\frac{|\mathrm{d}\lambda|}{|\lambda|} + \frac{2M}{2\pi}\int_{\gamma_+} |\lambda|^{-\beta}\,\frac{|\mathrm{d}\lambda|}{|\lambda|}
\end{aligned}
$$



Proof III

$$\begin{aligned}
\|F_n(hA)\| &\leq \frac{CM}{2\pi}\int_{\gamma_-}|\lambda|^{p+1-\beta}\,ne^{-cn|\lambda|}\,\frac{|\mathrm{d}\lambda|}{|\lambda|}+\frac{2M}{2\pi}\int_{\gamma_+}|\lambda|^{-\beta}\,\frac{|\mathrm{d}\lambda|}{|\lambda|}\leq\\
&\leq \frac{2CM}{2\pi}\int_0^1 s^{p-\beta}ne^{-cns}\,\mathrm{d}s+\frac{4M}{2\pi}\int_1^\infty s^{-\beta+1}\,\mathrm{d}s\leq\\
&\leq \frac{2CM}{2\pi}\int_0^\infty s^{p-\beta}ne^{-cns}\,\mathrm{d}s+\frac{4M}{2\pi}\int_1^\infty s^{-\beta+1}\,\mathrm{d}s\leq\\
&\leq \frac{2CM}{2\pi}\int_0^\infty s^{p-\beta}e^{-cs}\,\mathrm{d}s+\frac{4M}{2\pi}\int_1^\infty s^{-\beta+1}\,\mathrm{d}s=K.
\end{aligned}$$



Proof IV

If we now fix $k \in \mathbb{N}$, $k > \beta$, and define the function

$$G_n(z) := \left(r(z)^n - e^{nz}\right)(1-z)^{-k} = F_n(z)(-z)^{\beta}(1-z)^{-k},$$

we can obtain

$$G_n(hA) = \left(r(hA)^n - e^{nhA}\right)\mathcal{R}(1,hA)^k = F_n(hA)(-hA)^{\beta}\mathcal{R}(1,hA)^k.$$

Then, for any f from $\mathcal{D}\left((-A)^{\beta}\right)$ we have

$$\left\|\left(r(hA)^n - e^{nhA}\right)f\right\| \leq \left\|F_n(hA)(-hA)^{\beta}f\right\| \leq Kh^{\beta}\left\|(-A)^{\beta}f\right\|. \quad \blacksquare$$

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[3] C. Lubich, O. Nevanlinna: *On resolvent conditions and stability estimates.* BIT 31 (1991), no. 2, 293–313. MR1112225 (92h:65145)